Lecture 12 02/26/2018

Maxwell Equations and Electrodynamics

The most general electromagnetic phenomena are described by the four Maxwell equations:

$$\vec{\nabla}\cdot\vec{D}=\rho \quad , \quad \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \quad , \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

The conservation of electric charge requires that $\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$.

This is respected by Maxwell equations:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=\vec{\nabla}\cdot\vec{J}+\vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}=\nabla\cdot\vec{J}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})=\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

As mentioned before, since $\vec{\nabla}\cdot\vec{B}=0$, we can write:

$$\boxed{\vec{B}=\vec{\nabla}\times\vec{A}} \quad (\vec{A}\text{: vector potential})$$

Then:

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})=-\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)\Rightarrow\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

This implies that:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi \Rightarrow \boxed{\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}} \quad (\Phi\text{: scalar potential})$$

In the static limit, $\frac{\partial \vec{A}}{\partial t} \to 0$ and we recover the relation $\vec{E} = -\vec{\nabla}\Phi$ that is valid for electrostatics.

An important point to note is that the vector and scalar potentials are not physical quantities, while the $\vec{E}$ and $\vec{B}$ field are. This can be seen as infinite configurations of Φ and $\vec{A}$ related by "gauge transformations" give rise to exactly the same $\vec{E}$ and $\vec{B}$ fields:

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi \quad , \quad \Phi \to \Phi' = \Phi - \frac{\partial \chi}{\partial t}$$

Here χ is an arbitrary function of $\vec{x}$ and t. We then have:

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times (\vec{\nabla}\chi)}_{0} = \vec{\nabla} \times \vec{A}$$

$$-\vec{\nabla}\Phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\Phi + \vec{\nabla}\left(\frac{\partial \chi}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t} + \frac{\partial}{\partial t}(\vec{\nabla}\chi) = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Therefore, $\vec{E}$ and $\vec{B}$ fields remain unchanged under any gauge transformation. In order to treat Φ and $\vec{A}$ as dynamical fields that uniquely determine $\vec{E}$ and $\vec{B}$, we have to remove the redundancy due to gauge transformations. This process is called "gauge fixing". By appropriated choice of the function χ, we may choose a gauge that is described by a differential equation involving Φ and $\vec{A}$. Two important gauges that are often used are the following:

(1) Lorenz gauge: $\vec{\nabla}\cdot\vec{A} + \dfrac{1}{c^2}\dfrac{\partial \Phi}{\partial t} = 0$.

(2) Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$.

We can find the dynamical equations that are obeyed by Φ and $\vec{A}$ in these gauges. Considering space without any material, we have from Maxwell equations:

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\underbrace{\vec{\nabla}\times\vec{A}}_{=\vec{B}}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \underbrace{\mu_0\varepsilon_0}_{=\frac{1}{c^2}}\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) \Rightarrow \boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)}$$

Also:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)}$$

In the Lorenz gauge, where $\vec{\nabla}\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0$, we have:

$$\boxed{\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}\quad,\quad \nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}}$$

The equations look very simple in this gauge (basically inhomogeneous wave equations), which makes it an attractive gauge.

Nevertheless, in radiation problems, the Coulomb gauge is typically more useful. In this gauge, where $\vec{\nabla}\cdot\vec{A}=0$, we have:

$$\boxed{\nabla^2\Phi=\frac{-\rho}{\varepsilon_0}\quad,\quad \nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\Phi}$$

The first equation implies that the scalar potential is transmitted everywhere instantaneously, or, with infinite velocity. Therefore, the Φ field is unphysical in the Coulomb gauge. On the other hand, the second equation implies that the $\vec{A}$ field is propagated at the speed of light. That equation can be further simplified by dividing $\vec{J}$ into the <u>transverse</u> and <u>longitudinal</u> parts:

$$\vec{J}=\vec{J}_t+\vec{J}_\ell\quad,\quad \vec{\nabla}\times\vec{J}_\ell=\vec{\nabla}\cdot\vec{J}_t=0$$

Writing $\vec{J}_\ell = -\vec{\nabla}\Psi$, we have:

$$\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\vec{J}_\ell = -\nabla^2\Psi \Rightarrow \Psi(\vec{x}) = \frac{1}{4\pi}\int \frac{\vec{\nabla}'\cdot\Psi(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau' \Rightarrow \vec{J}_\ell(\vec{x},t) =$$

$$-\frac{1}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d\tau' = \frac{1}{4\pi}\vec{\nabla}\int \frac{\partial}{\partial t}\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d\tau' = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\vec{\nabla}\int \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d\tau'\right)$$

In free space with no boundary, the equation $\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$ has the usual solution:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\,d\tau'$$

Then:

$$\vec{J}_\ell(t) = \varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\Phi) = \frac{1}{c^2\mu_0}\frac{\partial}{\partial t}(\vec{\nabla}\phi) \Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0(\vec{J}-\vec{J}_\ell)$$

$$\Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}_t$$

This implies that the source of the vector potential $\vec{A}$ in the Coulomb gauge is the transverse part of the current density $\vec{J}$.